MOMENTUM AND ENERGY IN THE SCHWARZSCHILD METRIC

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ABSTRACT

Albert Einstein validated his field equations by demonstrating that they complied with what he called the laws of momentum and energy. The most well-known solution to Einstein's field equations is the Schwarzschild metric describing the gravitational field of a mass point. Here is examined how what Einstein called the laws of momentum and energy are manifest in the Schwarzschild metric and how these laws limit the geometry of space-time that is defined by the Schwarzschild metric.

INTRODUCTION

The laws of momentum and energy that underlie Einstein's field equations are by necessity incorporated into any solution of the field equations. Here is explored how momentum and energy is manifested in the Schwarzschild metric.

When there is no gravity field present, the Schwarzschild metric reduces to the Minkowski metric. The Minkowski metric is used to explore the energy and momentum resulting from motion in space and time apart from the presence of gravity. For the case where there is a gravity field present, but no motion through space, the Schwarzschild metric reduces to what is called herein a "no motion" metric. The no motion metric is used to explore the energy and momentum resulting from gravity apart from the presence of motion in

Doug Weller

space. The full Schwarzschild metric is used to explore the energy and momentum resulting from the presence of both gravity and motion in space.

The discussions on the Minkowski metric, the no motion metric and the full Schwarzschild metric each include one or more subsections showing how conservation of momentum and energy necessarily results in limiting the geometry of space-time that is described by each metric.

I. THE MINKOWSKI METRIC

The Schwarzschild metric describes the gravity field surrounding a point mass. Where the effect of gravity vanishes, the Schwarzschild metric reduces to the Minkowski metric.[1] Momentum and conservation are first examined in this simple case of the Schwarzschild metric.

The Minkowski metric was originally derived based on Hermann Minkowski's fundamental axiom for space-time set out in an address[2] given in September 1908:

> *The substance at any world-point may always, with the appropriate determination of space and time, be looked upon as at rest.*

Minkowski's fundamental axiom for the space-time continuum indicates that for the substance at a world point (e.g., a particle) there exists a local reference frame, with its own local space and time coordinates, in which the substance is at rest with respect to the local space coordinates (but not with respect to the local time coordinate).

Doug Weller

For example, assume the local reference frame for a particle has the local space

coordinates (ξ, η, ς) and the local time coordinate τ. For the particle, with respect to the

local reference frame,

$$\frac{d\xi}{d\tau} = \frac{d\eta}{d\tau} = \frac{d\varsigma}{d\tau} = 0. \tag{1}$$

The Minkowski metric provides information necessary to make a coordinate

transformation from the coordinates defining the local reference frame to reference

coordinates (*x, y, z, t*) defining another reference frame.

The Minkowski metric often appears in Cartesian coordinates as,

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2, \tag{2}$$

 arranged to provide information useful to obtain values of the time coordinate of the local

reference frame from values of the reference coordinates (*x, y, z, t*). The Cartesian coordinates

used to express the Minkowski metric can also be converted to spherical coordinates so that

the Minkowski metric has the form

$$c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - \left(r^2 \sin^2 \theta\right) d\varphi^2. \tag{3}$$

A. Selection of a reference frame from which to evaluate momentum and energy

In order to obtain information about momentum and energy within the Minkowski

metric (and the Schwarzschild metric) it is important to select and consistently use a reference

frame from which to make measurements. In the Minkowski metric there are two reference

frames to choose from. The first is the local reference frame defined by local coordinates

Doug Weller

(ξ,η,ς,τ). The other is the reference frame (referred to herein as the coordinate reference frame) defined by reference coordinates (x, y, z, t).

There is a distinct disadvantage to use of the local reference frame to make measurements: in its own local reference frame an object is always at rest, that is, as indicated by equation (1) there is no spatial velocity, i.e., no change in the values of the local space coordinates (ξ,η,ς) with respect to passage of time as measured by the time coordinate τ. When there is no motion through space, it is very difficult to evaluate momentum and kinetic energy.

In the coordinate reference frame, however, there can be a detectable motion through the space coordinates. This is referred to herein as spatial velocity ($\vec{v}_S$), which is a vector sum of the motion in three dimensions of space, i.e.,

$$\vec{v}_S = \vec{v}_x + \vec{v}_y + \vec{v}_z, \tag{4}$$

and which has a magnitude v_S where

$$v_S = |\vec{v}_S| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}, \tag{5}$$

as measured by the coordinate reference frame.

Because of this distinct advantage of making measurements from the coordinate reference frame, this is the reference frame that will be consistently used herein to make measurements.

Doug Weller

B. Detecting the expression of momentum and energy in the Minkowski Metric

The Minkowski metric, shown in equation (2) is organized in a form that provides information useful to obtain values of the time coordinate τ of the local reference frame from values of the reference coordinates (x, y, z, t). In order to obtain useful information about momentum and energy, it is helpful to mathematically reorganize the Minkowski metric to make this information more apparent.

Since the observer is making measurements from the coordinate reference frame, momentum and energy will need to be measured with respect to changes in the reference time coordinate t. The Minkowski metric is therefore rearranged to show this. Specifically, equation (2) can be rearranged as

$$c^2 dt^2 = c^2 d\tau^2 + dx^2 + dy^2 + dz^2, \tag{6}$$

and therefore,

$$c^2 = \left(c\frac{d\tau}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2, \tag{7}$$

which can be reduced to

$$c^2 = \left(c\frac{d\tau}{dt}\right)^2 + v_S^{\,2}. \tag{8}$$

The term $c\dfrac{d\tau}{dt}$ is a measure of the rate of passage of time as measured by the local time coordinate τ with respect to the rate of the passage of time as measured by the reference

Doug Weller

time coordinate t. The term $c\dfrac{d\tau}{dt}$ is therefore a measure of the velocity of local time with respect to coordinate time and is referred to herein as time velocity v_τ, where

$$v_\tau = c\frac{d\tau}{dt}. \tag{9}$$

This allows equations (7) to be rewritten as

$$c^2 = v_\tau{}^2 + v_S{}^2. \tag{10}$$

Since the time dimension is regarded as being orthogonal to the space dimensions equation (10) can be written in the form of a vector sum, i.e.,

$$c = \left|\vec{v}_S + \vec{v}_\tau\right|. \tag{11}$$

Equation (11) provides a very clear description of four-dimensional momentum in the Minkowski metric. That is, the vector sum of the velocity in the dimensions of time and space is always equal to the speed of light c.

Einstein's field equations, the Schwarzschild metric and the Minkowski metric all describe distribution of momentum and energy for matterless space.[3] In order to describe momentum and energy in more familiar terms, a particle with mass m can be placed in the local reference frame, in which case—from equation (11)—the momentum of particle m across the four dimensions of time and space can be expressed as

$$mc = \left|m\vec{v}_\tau + m\vec{v}_S\right|. \tag{12}$$

Equation (10) can also be rewritten as

$$mc^2 = mv_\tau{}^2 + mv_S{}^2, \tag{13}$$

Doug Weller

which indicates how the energy equivalence E of mass m, i.e.,

$$E = mc^2, \tag{14}$$

is apportioned by the Minkowski metric into an energy component E_τ in the time dimension, where

$$E_\tau = mv_\tau{}^2, \tag{15}$$

and an energy component in the space dimensions E_S in the space dimensions, where

$$E_S = mv_S{}^2, \tag{16}$$

so that

$$E = mc^2 = E_\tau + E_S. \tag{17}$$

In order to verify the concept of energy equivalence being apportioned into energy components, the relationship between kinetic energy and energy equivalence is explored in the following subsections.

C. Kinetic Energy

As made clear by the relationship between spatial velocity and time velocity v_τ set out in equation (7), when $v_S = 0$, then $v_\tau = c$. When particle m is moving in space with respect to reference space coordinates (x, y, z) there is a change in momentum from the rest state not only in the space component of momentum $m\vec{v}_S$ but also in the time component of momentum $m\vec{v}_\tau$. From rest to any space velocity v_S, there is a change in the value of the space component of momentum from 0 to the value mv_S in the direction of travel through

space. That is the change in momentum, $mv_S - 0 = mv_S$, is the value Newtonian Physics recognizes as the momentum of particle m.

When particle m moves from rest to any space velocity v_S, there is a change in the value of the time component of momentum from mc to the value mv_τ in the direction of time. This value for change in momentum $mc - mv_\tau$, is the momentum in the time dimension that is "sacrificed" to achieve space velocity v_S and which is restored to the time dimension when the space velocity is returned to 0.

From equation (10),

$$v_\tau = \sqrt{c^2 - v_S^2} , \tag{18}$$

so that

$$mc - mv_\tau = m\left(c - \sqrt{c^2 - v_S^2}\right). \tag{19}$$

For the case where $c \gg v_S$, a good binomial approximation is made using just the first two terms of the Taylor expansion so that

$$\sqrt{c^2 - v_S^2} \approx c - \frac{v_S^2}{2c}. \tag{20}$$

Therefore

$$mc - mv_\tau \approx mc - m\left(c - \frac{v_S^2}{2c}\right) = m\frac{v_S^2}{2c} , \tag{21}$$

which can be rewritten as

$$mc^2 - mcv_\tau = c(mc - mv)_\tau \approx \frac{1}{2}mv_S^2 = E_K . \tag{22}$$

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Equation (20) shows that the speed of light times the loss of momentum in the time dimension is equal to the value Newtonian Physics recognizes as the kinetic energy (E_K) of particle m.

D. Mass Energy Equivalence

In 1905, Albert Einstein[4] derived an energy equivalent for mass by calculating the difference in energy (ΔL) between light in a local reference frame defined by local space coordinates (ξ, η, ς) and the same light in the reference frame defined by space coordinates (x, y, z). The difference in energy ΔL was set equal to the Newtonian value for kinetic energy of a mass to produce Einstein's value for mass-energy equivalence.

To obtain ΔL, Einstein defined a value L to represent energy of light in the local reference frame defined by local coordinates (ξ, η, ς). Using the principle of the constancy of light, Einstein calculated the difference in energy (ΔL) to be

$$\Delta L = L\left(\frac{1}{\sqrt{1 - v_S^2/c^2}} - 1\right). \tag{23}$$

Einstein assumes that $c \gg v_S$ ("neglecting magnitudes of fourth and higher orders"), and thus simplifies equation (23) to

$$\Delta L \approx \frac{1}{2}\frac{L}{c^2}v_S^2. \tag{24}$$

Einstein compares this value to the Newtonian value for kinetic energy (E_K),

$E_K = \frac{1}{2}mv_S^2$ to derive the energy equivalence of mass. That is, $\Delta L = E_K$ so that

$$\frac{1}{2}\frac{L}{c^2}v_S^2 = \frac{1}{2}mv_S^2 \tag{25}$$

Doug Weller

Solving equation (25) for L, Einstein obtained $L = mc^2$ the equation for the energy

equivalence of mass, normally written in the form $E = mc^2$.

E. Calculating Mass-Energy Equivalence without using Approximations

Einstein's 1905 paper, written in the context of Newtonian physics, utilizes two

"Newtonian" approximations. The approximation represented by equation (24) is used to

calculate the difference in energy of light between two time frames. Einstein also implicitly

utilizes the approximation represented by equation (20) to obtain the Newtonian value for

kinetic energy resulting from the motion of a mass.

Einstein's method of calculating mass-energy equivalence can also be performed

without approximations. In this case, the value for E_K, calculated from equation (19) is

$$E_K = mc^2 - mcv_\tau = mc^2 - mc\sqrt{c^2 - v_S^2} = mc^2\left(1 - \sqrt{1 - v_S^2/c^2}\right). \tag{26}$$

Setting, as did Einstein, $E_K = \Delta L$, and using equation (26) to give the value for E_K and

equation (23) to give the value for ΔL yields

$$\Delta L = E_K \Rightarrow L\left(\frac{1}{\sqrt{1 - v_S^2/c^2}} - 1\right) = mc^2\left(1 - \sqrt{1 - v_S^2/c^2}\right). \tag{27}$$

Simplifying this equations leads to

$$L\left(\frac{1 - \sqrt{1 - v_S^2/c^2}}{\sqrt{1 - v_S^2/c^2}}\right) = mc^2\left(1 - \sqrt{1 - v_S^2/c^2}\right), \tag{28}$$

and

Doug Weller

$$L = mc^2 \left(\frac{1 - \sqrt{1 - v_S^2/c^2}}{1 - \sqrt{1 - v_S^2/c^2}} \right) \sqrt{1 - v_S^2/c^2} \,, \tag{29}$$

and

$$L = mc\sqrt{c^2 - v_S^2} \,, \tag{30}$$

and finally

$$L = mcv_\tau = mc^2 \frac{d\tau}{dt} \,, \tag{31}$$

Recall that according to Einstein's definition, L represents the energy equivalence of light in the local reference frame. When the local reference frame is the same as the coordinate reference frame, then

$$d\tau = dt \,, \tag{32}$$

and therefore

$$\frac{d\tau}{dt} = \frac{dt}{dt} = 1, \tag{33}$$

so that

$$L = mc^2 \,, \tag{34}$$

the same value obtained by Einstein, which is now typically expressed as $E = mc^2$.

E. Apportioning Mass-Energy Equivalence

As discussed above, Einstein calculated mass-energy equivalence based on the premise that the energy equivalence of light and thus the energy equivalence of matter, varies based on the reference frame in which the light or mass is located. When the mass is at rest in

Doug Weller

the local reference frame, than as measured by the coordinate reference coordinates, the total

energy equivalence E can be apportioned as

$$E = mc^2 = mcv_\tau + (mc^2 - mcv_\tau). \tag{35}$$

From equation (31), the energy component mcv_τ represents the equivalent energy E_L

of mass m in the local reference frame. From equation (22) the energy component

$(mc^2 - mcv_\tau)$ represents kinetic energy E_K, so that

$$E = mc^2 = E_L + E_K. \tag{36}$$

Equations (35) and (36) demonstrate the validity of apportioning energy equivalence E

of a mass m based on the value of time velocity v_τ, and therefore confirms equations (13) and

(17) fairly indicate the way the Minkowski metrics apportions energy equivalence E based on

the value of time velocity v_τ.

F. Discontinuity in the Minkowski metric

From equations (10) and (11), it is clear that the geometry of space defined by the

Minkowski metric is limited to regions where $v_S < c$. There is a time singularity (i.e.,

$v_\tau = 0 \Rightarrow \dfrac{d\tau}{dt} = 0$) in the Minkowski metric when $v_S = c$ and the Minkowski metric is

discontinuous when $v_S > c$.

To understand the physical reason for the time singularity, consider equation (35),

showing the apportionment of energy equivalence based on kinetic energy. When $v_\tau = 0$ the

entire energy equivalence $E = mc^2$ is used up by the kinetic energy component

$E_K = (mc^2 - mcv_\tau) = (mc^2 - mc(0)) = mc^2$. There is no available source of energy to be

further apportioned into kinetic energy. Kinetic energy has reached a maximum value.

Equations (12) and (13) are alternative ways of expressing this physical reason behind

the discontinuity in the Minkowski metric. For example, according to equation (12) when the

spatial velocity $v_S = c$, the entire momentum of mass m is used up by velocity in the space

dimension. There is no available momentum to be used for movement in the time dimension,

so time stops progressing. Likewise, in equation (13) when the spatial velocity $v_S = c$, the

entire energy equivalence $E = mc^2$ is used up by the energy component $E_S = mv_S{}^2$. There is

no available source of energy any increase in the value of the energy component E_S. Energy

component E_S has reached a maximum value.

The limitation $v_S \leq c$ is a well-known physical boundary that is readily seen when

evaluating momentum and energy in the Minkowski metric from the perspective of the

coordinate reference frame, but is not as evident when evaluating momentum and energy from

the perspective of the local reference frame. That is, as set out in equation (1), the particle is

stationary when measured using the local space coordinates (ξ,η,ς) so there is no momentum

in the space dimensions and there is no kinetic energy that is detectable using the local space

coordinates. The discontinuity is therefore detectable from the local reference system only

from the coordinate transformation to the coordinate reference frame. That is, when $v_S = c$,

there is a singularity in the relationship between the reference time coordinate and the local

time coordinate (i.e., $\dfrac{dt}{d\tau} = \infty$) which can be detected from the local space coordinates. In the

local reference frame the time singularity $\dfrac{dt}{d\tau} = \infty$ manifests as the momentum of coordinate time increasing to infinity.

Because the Minkowski metric is discontinuous for velocities greater than $v_S > c$, any values obtained from the Minkowski metric become nonsensical when $v_S > c$. For example, in equation (2), when $v_S > c$, $d\tau^2$ becomes negative. In attempt to make sense of this, Minkowski introduced the equation $t\sqrt{-1} = s$ to produce what he called the "mystic formula"[5]

$$3x10^5 \, km = \sqrt{-1} \, \sec .\qquad\qquad(37)$$

I. NO MOTION METRIC

The Schwarzschild metric reduces to the Minkowski metric where there is no gravity field present. This facilitated evaluation of how, in the Schwarzschild metric, momentum and energy is affected by motion in the absence of a gravity field.

The Schwarzschild metric can also be reduced to a "no motion" metric when the local reference frame is stationary in space with respect to the coordinate reference frame. The no motion metric facilitates evaluation of how, in the Schwarzschild metric, momentum and energy are affected by gravity in the absence of motion through space.

The full Schwarzschild metric for a point mass M with a Schwarzschild radius R, is typically expressed with the reference coordinates in the form of spherical coordinates, i.e.,

$$c^2 d\tau^2 = c^2(1 - \frac{R}{r})dt^2 - \frac{dr^2}{(1 - R/r)} - r^2 d\theta^2 - \left(r^2 \sin^2 \theta\right)d\varphi^2 .\qquad\qquad(38)$$

Where there is no motion through space, $dr = d\theta = d\varphi = 0$ and the Schwarzschild metric

reduces to the following "no motion" metric:

$$d\tau^2 = (1 - \frac{R}{r})dt^2.$$ (39)

In equation (39) the effects of gravity are taken into account by the quantity $\frac{R}{r}dt^2$,

which appears in equation (39) because when Karl Schwarzschild solved Einstein's field

equation to produce the Schwarzschild metric, he calculated components of the gravitational

field using a gravitational constant, as is done in Newtonian physics.[6] As a result, the

Schwarzschild metric accounts for the affect of the gravity using the Schwarzschild radius R

or in an equivalent form that instead includes the gravitational constant G, where

$$R = \frac{2GM}{c^2}.$$ (40)

Expressing the coordinates in the form of spherical coordinates rather than Cartesian

coordinates has no effect on calculated values of momentum and energy because the

coordinate reference frame does not change when the form of the coordinates change from

Cartesian to spherical coordinates. For example, where the origin for the Cartesian coordinate

system is located at the center of the point mass, the no motion metric can be expressed as

$$d\tau^2 = (1 - \frac{R}{\sqrt{x^2 + y^2 + z^2}})dt^2.$$ (41)

The same values for momentum and energy will be calculated whether equation (39)

or equation (41) is used. However, because of the shape of the gravity field, calculations

usually appear in a simpler form when using spherical coordinates. So spherical coordinates will be used in the remainder of the paper.

A. Detecting the expression of momentum and energy in the no motion metric

As when evaluating momentum and energy in the Minkowski metric, and for the same reasons, measurements of momentum and energy are made from the coordinate reference frame.

To facilitate this, the no motion metric expressed in equation (39) can be rearranged as

$$c^2 d\tau^2 = c^2 dt^2 - c^2 \frac{R}{r} dt^2, \tag{42}$$

and

$$c^2 dt^2 = c^2 d\tau^2 + c^2 \frac{R}{r} dt^2, \tag{43}$$

and

$$c^2 = c^2 \left(\frac{d\tau}{dt}\right)^2 + c^2 \frac{R}{r}, \tag{44}$$

and

$$c^2 = v_\tau^2 + c^2 \frac{R}{r}. \tag{45}$$

In order to highlight the similarities of the Minkowski metric to the no motion metric, it is possible to use the Newtonian definition of gravitational escape velocity v_G, that is

$$v_G = c \sqrt{\frac{R}{r}}, \tag{46}$$

to express equation (45) as

$$c^2 = v_\tau^{\;2} + v_G^{\;2}. \tag{47}$$

Equation (47) describes apportionment of energy in matterless space. In order to describe energy in more familiar terms, the particle with mass m can be placed in the local reference frame, so that

$$mc^2 = mv_\tau^{\;2} + mv_G^{\;2}, \tag{48}$$

which indicates how the energy equivalence E of mass m, i.e.,

$$E = mc^2, \tag{49}$$

is apportioned by the no motion metric into time energy component E_τ, where

$$E_\tau = mv_\tau^{\;2}, \tag{50}$$

and a gravitational energy component E_G, where

$$E_G = mv_G^{\;2}, \tag{51}$$

so that

$$E = mc^2 = E_\tau + E_G. \tag{52}$$

In accordance with parallel equations (where v_G replaces v_K) to those describing kinetic energy, gravitational potential energy E_P in the no motion metric can be defined as

$$E_P = (mc^2 - mcv_\tau). \tag{53}$$

allowing the total energy equivalence E in the no motion metric to be apportioned as

$$E = mc^2 = mcv_\tau + (mc^2 - mcv_\tau), \tag{54}$$

so that

$$E = mc^2 = E_L + E_P. \tag{55}$$

Doug Weller

Momentum in the no motion metric can be modeled by treating the time dimension as being orthogonal to gravitational escape velocity, so that equation (47) can be written in the form of a vector sum, i.e.,

$$c = \left| \vec{v}_G + \vec{v}_\tau \right|. \tag{56}$$

Equation (56) provides a description of momentum in the matterless space described by no motion metric. That is, vector sum of the velocity of time and the gravitational escape velocity is always equal to the speed of light c.

The particle with mass m can be placed in the local reference frame, in which case the momentum of particle m can be expressed as

$$mc = \left| m\vec{v}_\tau + m\vec{v}_G \right|. \tag{57}$$

F. Discontinuity in the no motion metric

From equation (56), it is clear that the geometry of space defined by the no motion metric is limited to regions where $v_G < c$ (i.e., $r>R$). There is a time singularity ($v_\tau = 0 \Rightarrow \dfrac{d\tau}{dt} = 0$) in the no motion metric when $v_G = c$ (i.e., $r=R$) and the no motion metric is discontinuous when $v_G > c$ (i.e., $r<R$).

To understand the physical reason for the time singularity, consider equation (55), showing the apportionment of energy equivalence based on gravitational potential energy. When $v_\tau = 0$ the entire energy equivalence $E = mc^2$ is used up by the gravitational potential energy component $E_P = (mc^2 - mcv_\tau) = mc^2$. There is no available source of energy to be

further apportioned into gravitational potential energy. Gravitational potential energy has reached a maximum value.

Equation (48) is an alternative way of expressing this physical reason behind the discontinuity in the no motion metric. According to equation (48) when the gravitational velocity $v_G = c$ (i.e., $r=R$), the entire energy equivalence $E = mc^2$ is used up by the energy component $E_G = mv_G^2$. There is no available source of energy any increase in the value of the energy component E_G. Energy component E_G has reached a maximum value.

At $r=0$, gravitational velocity v_G and thus energy component E_G blows up to infinity, indicating infinite energy would be required to, from space, reach the very center of mass M.

G. Effect of the Discontinuity on light

The no motion metric can be used to measure the progress of light as it travels radially towards a mass M to reach the Schwarzschild radius of a mass M. Measured from any reference frame, light fails to reach the Schwarzschild radius before mass M evaporates due to Hawking radiation.[7]

For example, consider the case of a series of local reference frames through which light passes on the way to the Schwarzschild radius R of mass M, each local reference frame having a local time coordinate τ. If there is no spatial motion between the reference frames, the no motion metric can be used for coordinate transformations between the reference frames. Therefore, from equation (39),

$$d\tau = dt\sqrt{1 - \frac{R}{r}}.\qquad(58)$$

Doug Weller

The radial coordinates for each reference frame can be used to measure the speed of light as it radially passes through the reference frame on the way to the Schwarzschild radius. That is, the radial coordinate r is used to measure the initial speed of light ($\frac{dr_L}{dt}$) at the location from where the light is initially transmitted from a light transmitter located at radial location r_T toward the Schwarzschild radius R. A local radial coordinate ρ is used to measure the speed of light ($\frac{d\rho_L}{d\tau}$) as it passes through each local reference frame. Since the speed of light as measured by local coordinates as it passes through a local reference frame is always equal to c, this means

$$\frac{dr_L}{dt} = \frac{d\rho_L}{d\tau} = c \,. \tag{59}$$

When measured by the coordinate reference coordinates, $\frac{dr_L}{dt} = c$ only at radial location r_T; therefore, in order to find the total time for light to reach the Schwarzschild radius it is necessary to find an integrand that takes into account how gravity affects the speed of light.

In order to preserve general relativity it is necessary that at every location length contraction is equal to the inverse of time dilation;[8] therefore,

$$\frac{dr_L}{d\rho_L} = \frac{d\tau}{dt} \,, \tag{60}$$

allowing equation (58) to be rewritten as

$$d\rho_L = \frac{dr_L}{\sqrt{1 - R/r}} \,. \tag{61}$$

Doug Weller

From equation (58) and equation (61),

$$\frac{d\rho_L}{d\tau} = \left(\frac{dr_L}{\sqrt{1-R/r}}\right) \Bigg/ \left(dt\sqrt{1-\frac{R}{r}}\right) = \frac{dr_L}{dt}\frac{1}{1-R/r}. \tag{62}$$

Combining equation (59) and equation (62) yields

$$c = \frac{dr_L}{dt}\frac{1}{1-R/r}. \tag{63}$$

and therefore

$$dt = \frac{dr_L}{c(1-R/r)}. \tag{64}$$

Equation (64) provides an integrand $\dfrac{dr_L}{c(1-R/r)}$ that can be used to calculate the time interval

Δt (i.e., coordinate travel time) for light to travel from a light transmitter located at any radial

location r_T, $r_T > R$, to the Schwarzschild radius R, i.e.,

$$\Delta t = \int_R^{r_T} dt = \int_R^{r_T} \frac{dr_L}{c\left(1-R/r\right)}. \tag{65}$$

The integral in equation (70) is divergent indicating time interval Δt is infinite. Since

mass M is not eternal—e.g., even a black hole will disintegrate eventually because of

Hawking radiation--the mass will evaporate before the light can reach it.[9] Assuming that

light from the light transmitter does not disintegrate first, there exists a radial location r_D

reached by the light before mass M evaporates where r_D>R.

Regardless of the radial starting location r_T, provided $r_T > R$, the divergence of the

integral in equation (65) indicates the time for light to travel to the Schwarzschild radius is

infinite. A convergent integral can be formed by using the series of local time coordinates

Doug Weller

through which the light passes to measure the total elapsed time for the light to make the journey from the location radial location r_T to the Schwarzschild radius R. The integrand can be obtained by rearranging equation (59) to

$$d\tau = \frac{d\rho_L}{c}.\tag{66}$$

and combining the result with equation (60),

$$d\tau = \frac{d\rho_L}{c} = \frac{dr_L}{c\sqrt{1 - R/r}}.\tag{67}$$

to obtain the integrand $\dfrac{dr_L}{c\sqrt{1 - R/r}}$. That is,

$$\Delta\tau_R = \int_R^{r_L} d\tau = \int_R^{r_L} \frac{dr_L}{c\sqrt{1 - R/r}}.\tag{68}$$

The convergent integral in equation (68) suggests that when measured using the local time coordinates, light can reach the Schwarzschild radius in finite time $\Delta\tau_R$.

However, the total elapsed time for the light to make the journey from the location radial location r_T to the radial location r_D reached by the light before mass M evaporates can also be calculated using the same integrand. That is,

$$\Delta\tau_D = \int_{r_D}^{r_T} d\tau = \int_{r_D}^{r_T} \frac{dr_L}{c\sqrt{1 - R/r}}.\tag{69}$$

Inevitably, $\Delta\tau_D < \Delta\tau_R$, which indicates that even when a convergent integral is used to calculate the time it takes light to travel to the Schwarzschild radius, the light still will not reach the Schwarzschild radius before the mass M evaporates. This result is predicted by the theory of general relativity. That is, general relativity predicts the laws of physics hold

Doug Weller

equally well even when measured from different reference frames, so that the same physical reality is observed from different reference frames.

II. THE COMPLETE SCHWARZSCHILD METRIC

Within the context of the Schwarzschild metric, the Minkowski metric describes momentum and energy when there is no gravity present and the no motion describes momentum and energy when there is no motion present. It would seem, therefore, very reasonable to conclude the Schwarzschild metric can be derived by a direct combination of the Minkowski metric and the no motion metric.

However, the Schwarzschild metric, as set out in equation (38) using the Schwarzschild coordinates as reference coordinates, has an additional multiplier in one of the terms. Specifically, in the radial term $\dfrac{dr^2}{(1-R/r)}$ of the Schwarzschild metric as set out in equation (38), there is a multiplier $\dfrac{1}{(1-R/r)}$ that is not contained in the Minkowski metric or the no motion metric. The existence of this multiplier is the reason it has been asserted that there is no simple derivation of the Schwarzschild metric.[10]

The multiplier in the radial term of the Schwarzschild metric makes the radial term a combination motion and gravity term. This combination motion and gravity term is usually treated as a curvature in space affecting the Schwarzschild metric spatial velocity v_{SS}, so that v_{SS} is defined in the full Schwarzschild metric as

$$v_{SS} = \sqrt{\frac{1}{(1-R/r)}\left(\frac{dr}{dt}\right)^2 - r^2\left(\frac{d\theta}{dt}\right)^2 - \left(r^2\sin^2\theta\right)\left(\frac{d\varphi}{dt}\right)^2} \ . \tag{70}$$

The Schwarzschild metric in equation (38) can be expressed in simpler terms using the definition for v_{SS} set out in equation (70). That is, equation (38) can be rearranged as

$$c^2dt^2 = c^2d\tau^2 + c^2\frac{R}{r}dt^2 + \frac{dr^2}{(1-R/r)} + r^2d\theta^2 + \left(r^2\sin^2\theta\right)d\varphi^2 , \tag{71}$$

and thus

$$c^2 = c^2\left(\frac{d\tau}{dt}\right)^2 + c^2\frac{R}{r} + \frac{1}{(1-R/r)}\left(\frac{dr}{dt}\right)^2 - r^2\left(\frac{d\theta}{dt}\right)^2 - \left(r^2\sin^2\theta\right)\left(\frac{d\varphi}{dt}\right)^2 . \tag{72}$$

Using the definition of v_{SS} set out in equation (70), the definition of v_τ set out in equation (9) and the definition of v_G set out in equation (46), allows equation (72) to be simplified to

$$c^2 = v_\tau{}^2 + v_G{}^2 + v_{SS}{}^2 . \tag{73}$$

A. Discontinuity in the Schwarzschild metric

From equation (73), the geometry of space defined by the full Schwarzschild metric is limited to regions where $c^2 > v_{SS}{}^2 + v_G{}^2$. There is a time singularity ($v_\tau = 0 \Rightarrow \dfrac{d\tau}{dt} = 0$) in the Schwarzschild metric when $c^2 = v_{SS}{}^2 + v_G{}^2$ and the Schwarzschild metric is discontinuous when $c^2 < v_{SS}{}^2 + v_G{}^2$.

Doug Weller

Equation (73) describes apportionment of energy in matterless space. In order to describe energy in more familiar terms, the particle with mass *m* can be placed in the local reference frame, so that

$$mc^2 = mv_\tau^2 + mv_{SS}^2 + mv_G^2 = mv_\tau^2 + mv_S^2 + mc^2 \frac{R}{r}, \tag{74}$$

which indicates how the energy equivalence *E* of mass *m*, i.e.,

$$E = mc^2, \tag{75}$$

is apportioned by the Schwarzschild metric into time energy component E_τ, where

$$E_\tau = mv_\tau^2, \tag{76}$$

a space energy component E_{SS}, where

$$E_{SS} = mv_{SS}^2, \tag{77}$$

and a gravitational energy component E_G, where

$$E_G = mv_G^2 = mc^2 \frac{R}{r}, \tag{78}$$

so that

$$E = mc^2 = E_\tau + E_{SS} + E_G. \tag{79}$$

B. Effect of the Discontinuity on travelers to the Schwarzschild radius

The integrals to determine exactly the amount of coordinate time and local time it takes for a traveler composed of matter to reach the Schwarzschild radius have been performed, or at least approximated, elsewhere.[11] The integral used to calculate coordinate time is divergent. The integral used to calculate local time is convergent. However, in both

coordinate reference time and in local time, any mass *M* compacted below its Schwarzschild radius to form a black hole will completely evaporate before the Schwarzschild radius can be reached.

Even without performing the integrals for the full Schwarzschild metric, it should be clear that regardless of the measure of time used, the black hole would completely evaporate before any traveler could reach the Schwarzschild radius. The time for the optimal traveler, light, to make the journey was previously calculated. Light is the fastest traveler and will therefore travel the farthest distance before the black hole evaporates. Since the light cannot traverse the distance to the Schwarzschild radius before a black hole evaporates, no slower moving traveler will be able to either, regardless of the time coordinate used to measure elapsed time for the journey.

However, it is highly improbable that a traveler could survive long enough to view the complete evaporation of a black hole. Consider the travails of a traveler to a black hole as observed by a distant observer. The distant observer will observe that during the lifetime of the black hole, background radiation travelling in a path that intersects the traveler will overtake the slower moving traveler. Such background radiation will continue to overtake and impact the distant observer throughout the entire lifetime of the black hole, or until the traveler disintegrates, whichever first occurs.

If the traveler can survive the bombardment of background radiation, bigger problems lie ahead. The distant observer observes the evaporation of the black hole before the Schwarzschild radius is reached by the traveler. The traveler, in an accelerated time frame, will experience the evaporation of the black hole as it is observed from a distance. The

Hawking radiation from the evaporating mass will first have to pass through the radial location of the traveler before reaching the distant observer. This insures the traveler will necessarily observe and experience, *before the distant observer*, radiation emitted during the disintegration of the mass. Further, the radiation passing by the traveler will continuously bring information to the distant observer about the location of the traveler. Each photon of radiation that passes by the traveler is a progress report on the traveler's location that will confirm to the distant observer that the traveler had not yet passed through the Schwarzschild radius when that photon of radiation passed the traveler. Such progress reports will continue until the black hole completely evaporates.

C. Implication for formation of black holes

Assuming that Einstein's field equations provide an accurate description of the laws of momentum and energy and that Schwarzschild metric is a solution to Einstein's field equations, then collapsing matter cannot compact below the Schwarzschild radius of a mass. To do so would violate the conservation of energy.

If the surface of a collapsing mass were to reach the Schwarzschild radius, particles of mass m on the surface would each have a gravitational energy component $E_G = mc^2$. The gravitational component would use up their entire energy equivalence $E = mc^2$. There is no source for the gravitational energy necessary for the particles to continue moving inward.

For the surface of a collapsing mass to reach *r=0* would require that particles of mass *m* on the surface each have a gravitational energy component $E_G = \infty$. The source of this infinite energy has never been identified.

D. Alternative Explanation for the end point of a compacting mass

Particles on the surface of a compacting mass are in essence travelers to the Schwarzschild radius of the mass. The closer the traveler comes to the Schwarzschild radius, the more accelerated the traveler's contact with radiation, both background radiation and radiation from the evaporation of fellow travelers. The temperature increase at the surface resulting from the presence of concentrated radiation, the pressure produced by extreme gravity and the shrinking volume would only serve to hasten the traveler's own evaporation into radiation. The result near the Schwarzschild radius would be an inferno of unimaginable proportions that would rapidly evaporate the surface of any mass compacted to a radius near its Schwarzschild radius.

Any evaporation of matter at the surface of the mass would tend to push the Schwarzschild radius down and away from the surface of the compacting mass, thereby insuring that the mass could never be compacted below its Schwarzschild radius. Specifically, as shown by equation (40), the Schwarzschild radius is proportional to mass *M*; however, radius *r* of the mass *M,* is related to the volume of mass *M* in accordance with the well known relationship

$$r = \sqrt[3]{V\frac{3}{4\pi}} \ . \qquad (80)$$

Assuming there is a rough correspondence between volume and mass, any evaporation on the surface of mass M, will reduce the Schwarzschild radius R much quicker than it will reduce radius r, as calculated in equations (40) and (80) respectively, with the result that the Schwarzschild radius R will shrink back down away from the surface of mass M, located at location r.

As observed from a distance, time dilation would tend to hide the intensity of the inferno at a surface near the Schwarzschild radius of a mass compacted to a radius near its Schwarzschild radius R; however, a rapid retreat of the Schwarzschild radius R from the surface of a mass as result of disintegrating matter at the surface would diminish the concealing effect of time dilation. The result would be the sudden release of a less time dilated view of the inferno occurring at the surface. Such sudden changes in time dilation provides a potential explanation for the sudden appearance of quasars.

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[5] Minkowski, §4.
[6] See K. Schwarzschild, §4 (Schwarzschild's integration constant α incorporates a gravitational constant).
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